hep-th/0407035
DAMTP-2004-71

A note on the uniqueness of the Neumann matrices in the plane-wave background

James Lucietti

DAMTP, Centre for Mathematical Sciences,
University of Cambridge, Wilberforce Rd.,
Cambridge CB3 0WA, UK
E-mail: J.Lucietti@damtp.cam.ac.uk

Abstract

In this note, we prove the uniqueness of the Neumann matrices of the open-closed vertex in plane-wave light-cone string-field theory, first derived for all values of the mass parameter μ in [1]. We also prove the existence and uniqueness of the inverse of an infinite dimensional matrix necessary for the cubic vertex Neumann matrices, and give an explicit expression for it in terms of μ-deformed Gamma functions. Methods of complex analysis are used together with the analytic properties of the μ-deformed Gamma functions. One of the implications of these results is that the geometrical continuity conditions suffice to determine the bosonic part of the vertices as in flat space.

The plane-wave limit of the AdS/CFT correspondence has provided a concrete arena for testing the validity of the duality. In this limit, we have a relation between string theory in the plane-wave limit of $AdS_5 \times S^5$ [2, 3] and a certain sector (BMN) of $\mathcal{N} = 4$, $d = 4$ super Yang-Mills (SYM) theory [4]. The beauty of this limit is that both the string theory and gauge theory side are perturbative.

Interestingly, this area has also motivated the development of certain deformations of classical special functions such as the Theta functions and the Gamma function. The deformed Theta functions appeared in expressions for the cylinder diagrams that determine the static interactions between pairs of Dp-branes in the type IIB plane-wave background [5, 6, 7]. The so-called "μ-deformed Gamma functions" first appeared, in several flavours, in the expressions for the Neumann matrices of the open-closed vertex [1] and the cubic vertex of plane-wave light-cone string field theory [8]. These Neumann matrices beautifully generalise the ones in flat-space obtained long ago [9, 10] in terms of the ordinary Gamma function. Other unexpected results in classical complex analysis have occurred from research in this area, such as a generalisation of an integral transform called the Stieltjes transform [11], which came about from the original derivation of the cubic vertex [12, 13].

In this letter, we provide a proof of the uniqueness of the Neumann matrices A_{mn}, B_{mn} and C_{mn}, which were determined in the solution to the bosonic part of the open-closed vertex [1]. We also provide a proof of existence and uniqueness of the inverse of a certain infinite dimensional matrix called Γ_+, from which we deduce uniqueness of the cubic vertex Neumann matrices $\bar{N}^{rs}_{mn}$, which were determined in [12, 8].

As a reminder, in [1] we solved[1] for the bosonic part of the vertex $|V\rangle_B$ using the geometrical continuity conditions

$$X^i(\sigma)_{open}|V\rangle_B = X^i(\sigma)_{closed}|V\rangle_B\,, \qquad [P^i(\sigma)_{open} + P^i(\sigma)_{closed}]|V\rangle_B = 0\,, \tag{1}$$

which are understood to hold at $\tau = 0$, the interaction time ($i = 1, ..., 8$ are the transverse coordinates). Working in terms of modes allows one to deduce an expression for the vertex entirely analogous to the one in flat space[2]

$$|V\rangle_B = \exp(\Delta)|0\rangle \tag{2}$$

where

$$\Delta = -\sum_{m=1}^{\infty} \frac{\sqrt{2}}{\omega_{2m}} \beta_{-2m}\alpha^I_{-m} + \sum_{m,n=0}^{\infty} A_{mn}\beta_{-2m-1}\alpha^{II}_{-n} + \frac{1}{2}B_{mn}\beta_{-2m-1}\beta_{-2n-1} + \frac{1}{2}C_{mn}\alpha^{II}_{-m}\alpha^{II}_{-n}$$

and $\alpha_n^{I/II} = \sqrt{2}(\alpha_n \pm \tilde{\alpha}_n)$, $\omega_n = \mathrm{sgn}\,(n)\sqrt{n^2+\mu^2}$ and here $|0\rangle$ is the vacuum of the two string Fock space. The modes α_n and $\tilde{\alpha}_n$ are of the closed string, and β_m of the open string, see [1] for the explicit modes expansions and commutators. The Neumann matrices A_{mn}, B_{mn} and C_{mn} satisfy the following complicated set of coupled equations:

$$-\frac{2\sqrt{2}in}{\pi} \sum_{m=0}^{\infty} \frac{1}{\omega_{2m+1} - \omega_{2n}} A_{mk} = \delta_{n,k} \tag{3}$$

$$-\sum_{m=0}^{\infty} \frac{B_{mk}}{\omega_{2m+1} - \omega_{2n}} = \frac{1}{(\omega_{2n} + \omega_{2k+1})\omega_{2k+1}} \tag{4}$$

$$\frac{4\sqrt{2}in}{\pi\omega_{2n}} \sum_{m=0}^{\infty} \frac{1}{\omega_{2m+1} + \omega_{2n}} A_{mp} = C_{np} \tag{5}$$

$$\frac{4\sqrt{2}in}{\pi\omega_{2n}} \left(\sum_{m=0}^{\infty} \frac{1}{\omega_{2m+1} + \omega_{2n}} B_{mp} + \frac{1}{(\omega_{2p+1} - \omega_{2n})\omega_{2p+1}} \right) = A_{pn}\,. \tag{6}$$

We found that a solution to this system of equations is given by

$$A_{mk} = i\sqrt{2}\frac{v^I_m v^{II}_k}{(\omega_{2m+1} - \omega_{2k})} \tag{7}$$

$$B_{mk} = \frac{v^I_m v^I_k}{(\omega_{2m+1} + \omega_{2k+1})} \tag{8}$$

$$C_{mk} = 2\frac{v^{II}_m v^{II}_k}{(\omega_{2m} + \omega_{2k})}\,. \tag{9}$$

These are written in terms of deformations of the binomial coefficients $u_n = \frac{\Gamma(n+1/2)}{\sqrt{\pi}\,\Gamma(n+1)}$ of $(1-x)^{-1/2} = \sum_{n=0}^{\infty} \frac{u_n x^n}{n!}$, which take the form

$$v^I_m = \frac{(2m+1)}{\omega_{2m+1}} \frac{\Gamma^I_\mu(m+1/2)}{\sqrt{\pi}\,\Gamma^{II}_\mu(m+1)} \tag{10}$$

$$v^{II}_m = \frac{2}{\omega_{2m}} \frac{\Gamma^{II}_\mu(m+1/2)}{\sqrt{\pi}\,\Gamma^I_\mu(m)}\,, \tag{11}$$

[1] We should note that the large μ asymptotics were first derived in [14] without the knowledge of the exact expression valid for all μ first obtained in [1].

[2] We will only concern ourselves with the case of Neumann boundary conditions for the open string. In [1] Dirichlet boundary conditions were also addressed.

where $\Gamma^I_\mu(z)$ and $\Gamma^{II}_\mu(z)$ are the μ-deformed Gamma functions of the first and second kind introduced in [1], which both reduce to $\Gamma(z)$ in the flat space limit $\mu \to 0$. One can find the definitions of these functions, together with some of their key properties in the Appendix.

In that paper a method was suggested for proving uniqueness of A_{mn}, B_{mn} and C_{mn}, at least in flat space, however this relied on some fairly heavy-handed complex analysis. It first involved determining the singularity structure of the matrices. This is the weak point of the technique, as this could only really be motivated, albeit strongly, and not proved. The method used here however, in light of the complicated nature of the set of equations which the Neumann matrices satisfy, is remarkably simple. It relies on an elementary observation of the property of inverses of infinite dimensional matrices. Namely, if both a right and left inverse exist then they are equal and *unique*. Note that for infinite dimensional matrices one can have the situation where we have multiple left inverses but no right inverses, for example (see Appendix D of [10]). Fortunately it so happens that one of the Neumann matrices, A_{mn}, is actually the right inverse of a certain infinite dimensional matrix. Thus a proof of uniqueness rests on showing that it is also the left inverse, which we prove in this letter.

In this letter we also prove existence and uniqueness of the Neumann matrices $\bar{N}^{rs}_{mn}$ of the cubic vertex. As a reminder [15, 16] the bosonic part of the vertex is written as

$$|V\rangle_B = \exp\left(\frac{1}{2}\sum_{r,s=1}^{3}\sum_{m,n=-\infty}^{\infty}\sum_{i=1}^{8} a^{i\dagger}_{rm}\bar{N}^{rs}_{mn}a^{i\dagger}_{sm}\right)|0\rangle \tag{12}$$

where a^i_{rm} are the modes of the three closed strings, normalised to satisfy the harmonic oscillator algebra $[a^i_{rm}, a^{j\dagger}_{sn}] = \delta_{mn}\delta^{ij}$ and $m \in \mathbb{Z}$ (note $a^\dagger_m \neq a_{-m}$ here). Here, the state $|0\rangle$ denotes the three string Fock vacuum (this satisfies $a^i_{rm}|0\rangle = 0$ for $m \in \mathbb{Z}$). The three strings have incoming momenta $\alpha_r \equiv 2p^{(r)+}$, so $\sum_{r=1}^3 \alpha_r = 0$. As in flat space we can "factorise" the Neumann matrices in terms of Neumann vectors $\bar{N}^r_m$ [17, 18], such that for $m, n \geq 1$ [3]

$$\bar{N}^{rs}_{mn} = -\frac{mn\alpha}{1+\mu\alpha k}\frac{\bar{N}^r_m\bar{N}^s_n}{\alpha_s\omega_{r,m}+\alpha_r\omega_{s,n}}, \tag{13}$$

where $\omega_{r,m} = \operatorname{sgn}(n)\sqrt{n^2+\alpha_r^2\mu^2}$. The Neumann vectors can be written as

$$\bar{N}^r_m = \sqrt{\frac{\omega_{r,m}}{m}}\frac{(\omega_{r,m}+\alpha_r\mu)}{m}\frac{1}{\alpha_r}f^{(r)}_m \tag{14}$$

and in [8] we determined the vectors $f^{(r)}_m$, and the scalar $k = B^T f^{(3)}$, by finding a solution to the two sums

$$\sum_{p=1}^{\infty} f^{(3)}_p A^{(r)}_{pm} = \frac{\alpha_3}{\alpha_r} f^{(r)}_m \tag{15}$$

$$\sum_{p=1}^{\infty}\sum_{r=1}^{3}\frac{1}{\alpha_r}\left(A^{(r)}U^{(r)}\right)_{mp} f^{(r)}_p = -B_m \tag{16}$$

which when combined give $\Gamma_+ f^{(3)} = B$; these sums can be derived from the appropriate geometrical continuity conditions. See the Appendix for the definitions of the matrices $A^{(r)}$, $U^{(r)}$ and the vector

[3] One can relate the Neumann matrices for the negative mode numbers to the ones with positive mode numbers, so it is sufficient to restrict to positive ones. In particular, the only non-vanishing matrix elements for negative modes are given by $\bar{N}^{rs}_{-m-n} = -(U^{(r)}\bar{N}^{rs}U^{(s)})_{mn}$.

B. The solutions we found are expressed in terms of μ-deformed Gamma functions as follows:

$$f_m^{(r)} = \frac{e^{\tau_0(\mu+\omega_{\frac{m}{\alpha_r}})}}{\sqrt{m}(-\alpha_r - \alpha_{r+1})\,\omega_{\frac{m}{\alpha_r}}} \frac{\Gamma_\mu^{(r+1)}\left(-\frac{m}{\alpha_r}\right)}{\Gamma_\mu^{(r)}\left(\frac{m}{\alpha_r}\right)\Gamma_\mu^{(r-1)}\left(\frac{m}{\alpha_r}\right)} M(0^+)\,, \tag{17}$$

and

$$k = \frac{1}{\alpha\mu}(M(0^+)^2 e^{2\tau_0\mu} - 1)\,. \tag{18}$$

See the Appendix for definitions of the Gamma functions and the constant $M(0^+)$. Note that an alternate, less direct method was first used in [12] to derive these quantities. To relate the notations of the two papers [12, 8], we have $Y = f^{(3)}$ and $k = -4K$, although for convenience here we will use k.

One might ask whether a simple proof of uniqueness could be presented for the Neumann matrices of the cubic vertex as well. As stressed, in [8] we found a solution $f^{(3)}$ to $\Gamma_+ f^{(3)} = B$. It is clear that if we could show that Γ_+ possesses a left inverse then the solution $f^{(3)}$ is indeed unique. At this point we note a welcome simplification: since Γ_+ is actually a symmetric matrix, existence of a left inverse implies existence of a right inverse, which in turn tells us that these inverses are equal and unique. Thus the problem of uniqueness in this case reduces to showing that Γ_+^{-1} actually exists! It appears that no-one has actually checked existence of this matrix. In [12] an expression for the inverse is derived[4] *assuming* it exists - of course to complete the proof one needs to verify that the expression is the inverse directly. This is possible using the methods of [1, 8]. We will present this calculation too, however it is rather more involved than than the calculation for the open-closed vertex. The great advantage of using the deformed Gamma functions in the expressions of the Neumann matrices is that the infinite dimensional matrix algebra required can be done explicitly just using elementary complex analysis. In fact what we will do is evaluate the infinite sum $\Gamma_+\Gamma_+^{-1}$ and confirm is it equal to the identity.

A proof of uniqueness is important in the plane-wave theory. It means that the geometrical continuity conditions imposed to derive the set of equations satisfied by the Neumann matrices, using Fock space methods, actually suffice to settle the bosonic part of the vertex. This is a nice result as it means that this method can probably be used more generally. For example, one might contemplate using geometrical continuity and the oscillator methods to compute the 1-loop correction to the closed string propagator, rather than directly "gluing" two cubic vertices together. Note that in flat-space one could deduce uniqueness by appealing to the alternate Green's function method of deriving the matrices. However, these methods, at the present at least, are absent in the plane-wave case due to the lack of explicit conformal invariance on the worldsheet. The proofs given here actually work in flat-space too (as they are valid for all μ!) and it appears that they have not been noticed before.

Finally, we should emphasise the advantages of our method as compared to the one used in [12]:

- It is not clear how one would generalise the technique used in [12] to other interaction vertices, whereas the techniques developed and used in [1, 8] are clearly more generally applicable as they allow one to solve the geometrical continuity conditions directly.
- Our method is a direct generalisation of the techniques which can be used in flat space.
- Expressing the Neumann matrices in terms of the deformed Gamma functions is not merely of aesthetic value. Once armed with their analytic properties it provides a powerful calculation tool.

[4]More precisely an expression for the vector Y and scalar k is derived which is enough information to deduce Γ_+^{-1} using an identity derived in [17].

Now we present our proofs. As we have already hinted at, the proof is based on the following observation. Consider the equation

$$\sum_m M_{nm} A_{mk} = \delta_{nk} \ , \tag{19}$$

which expresses the fact that A is the right-inverse of M when they are interpreted as infinite dimensional matrices. If A is also the left-inverse to M then the solution (i.e. solving for A when M is known) to (19) is unique. This follows since we have $\sum_k A_{mk} M_{kn} = \delta_{mn}$, and thus if there exists another solution to (19), say A', then we have $A(MA') = A$ which implies $A' = A$ by associativity [5].

In solving for the open-closed vertex we came across the equation,

$$\sum_{m=0}^{\infty} M_{nm} A_{mk} = \delta_{nk} \ , \tag{20}$$

where

$$M_{nm} = \frac{i2\sqrt{2}n}{\pi(\omega_{2n} - \omega_{2m+1})} \ , \tag{21}$$

and

$$A_{mk} = \frac{i\sqrt{2} v^I_m v^{II}_k}{\omega_{2m+1} - \omega_{2k}} \ . \tag{22}$$

Therefore, based on the above observation if A_{mn} is also the left-inverse to M_{mn} then it is the unique solution to (20). Thus we need to prove the following sum

$$\sum_{k=1}^{\infty} \frac{A_{mk} 2\sqrt{2}ik}{\pi(\omega_{2k} - \omega_{2n+1})} = \delta_{mn} \ . \tag{23}$$

To this end consider the contour integral

$$\oint \frac{dk}{2\pi i} \pi \cot(\pi k) A_{mk} \frac{2\sqrt{2}ik}{\pi(\omega_{2k} - \omega_{2n+1})} \ . \tag{24}$$

Recall $v^{II}_k = \frac{2\Gamma^{II}_\mu(k+1/2)}{\Gamma^I_\mu(k)\omega_{2k}\sqrt{\pi}}$, which tells us that A_{mk} has zeroes for $k \in \mathbb{N}_0$, simple poles at $k = -1/2, -3/2, -5/2, \ldots$ and another simple pole at $k = m + 1/2$. A_{mk} also has a branch cut on $[i\mu/2, -i\mu/2]$ and branch points at $\pm i\mu/2$. Therefore if $n \neq m$ the integrand in (24) only has poles at $k = 1, 2 \ldots$ (the rest are cancelled), in which case the remaining contributions come from the branch cuts, branch points and the integral at infinity. Using the asymptotics of v^{II}_k we see the integrand goes as $k^{-3/2}$ and thus the integral at infinity does not contribute. Since $v^{II}_{0^\pm + iy}$ is odd for $|y| < \mu/2$, the integrand on either side of the branch cut is odd and thus these contributions vanish too. Finally the integrals around the branch points vanish as $O(\epsilon^{1/2})$, where ϵ is the radius from the branch point. Thus we have verified (23) for $n \neq m$. So now we consider $n = m$; the argument proceeds in the same way except now the integrand also has a simple pole at $k = n + 1/2$. Thus we also need to include the following residue

$$\begin{aligned} &\mathrm{Res}_{k=n+1/2}\, \pi \cot(\pi k) A_{nk} \frac{2\sqrt{2}ik}{\pi(\omega_{2k} - \omega_{2n+1})} \\ &= 2(2n+1) v^I_{n+1/2} v^{II}_n \lim_{k \to n+1/2} \frac{\cot(\pi k)}{\omega_{2k} - \omega_{2n+1}} \mathrm{Res}_{k=n+1/2} \frac{1}{\omega_{2k} - \omega_{2n+1}} = -1 \ , \end{aligned} \tag{25}$$

[5] Associativity of matrix multiplication in the infinite dimensional case is equivalent to swapping the order of two infinite sums and thus is related to the convergence properties of the sums in question.

which now proves (23). We deduce that A_{mn} is unique and we are entitled to call $M_{mn} = A^{-1}_{mn}$. Note it immediately follows that C_{mn} is unique too, as it is expressed entirely in terms of A_{mn}, see (5). To prove that B_{mn} is also unique, we note that one of the equations satisfied by B_{mn} is

$$\sum_{m=0}^{\infty} M_{nm} B_{mk} = \frac{2\sqrt{2}in}{\pi\omega_{2k+1}(\omega_{2k+1}+\omega_{2n})} \ . \tag{26}$$

Therefore multiplying (26) by A_{pn} and summing from $n = 1, ..., \infty$, and using (23) (i.e. using the fact that A is left-inverse to M) implies

$$B_{mk} = \sum_{n=1}^{\infty} \frac{2\sqrt{2}inA_{mn}}{\pi\omega_{2k+1}(\omega_{2n}+\omega_{2k+1})} \ . \tag{27}$$

It follows that B_{mn} is also unique, thus proving the theorem. Note, one could of course check that the RHS of (27) reproduces the correct B_{mn} by doing the sum in the usual manner. Also note the uniqueness theorem follows for flat space too by setting $\mu = 0$.

A crucial ingredient of the Neumann matrices of the cubic vertex, is the inverse of a certain infinite dimensional matrix Γ_+. It is defined as

$$\Gamma_+ = \sum_{r=1}^{3} A^{(r)} U^{(r)} {A^{(r)}}^t \tag{28}$$

and one can find the definitions of the matrices $A^{(r)}$ and $U^{(r)}$ in the Appendix. As already explained, there is a lose end to tie up here. In [12], the existence of the inverse was assumed and using this an expression for it was derived. However no-one has checked that the final explicit expression for Γ_+^{-1} actually is the inverse! In [8] we found a vector $f^{(3)}$ which satisfied $\Gamma_+ f^{(3)} = B$. However we did not manage to prove it was unique. Here we will settle both of these open ends by taking the explicit expression for Γ_+^{-1} in [12] and multiplying it into Γ_+. The existence of an inverse ensures it is the unique left and right inverse (due to the matrix being symmetric), and also establishes that the solution in [8] is unique.

Thus, the candidate expression for Γ_+^{-1} is [17]

$$(\Gamma_+^{-1})_{mn} = \frac{m}{2\omega_{3,m}}\delta_{mn} + \frac{\alpha_1\alpha_2(\omega_{3,m}+\mu\alpha_3)(\omega_{3,n}+\mu\alpha_3)f_m^{(3)}f_n^{(3)}}{2(\omega_{3,m}+\omega_{3,n})(1+\mu\alpha k)} \ . \tag{29}$$

The sum we are interested in is

$$\sum_{p=1}^{\infty}(\Gamma_+)_{mp}(\Gamma_+^{-1})_{pn} = \frac{m}{2\omega_{3,m}}(\Gamma_+)_{mn} + \frac{\alpha_1\alpha_2(\omega_{3,n}+\mu\alpha_3)f_n^{(3)}}{2(1+\alpha\mu k)}\sum_{p=1}^{\infty}(\Gamma_+)_{mp}f_p^{(3)}\frac{\omega_{3,p}+\mu\alpha_3}{\omega_{3,p}+\omega_{3,n}}. \tag{30}$$

Evaluation of the RHS of (30) can be performed using two identities which we will prove. We will work in the gauge $\alpha_1 = y$, $\alpha_2 = 1-y$ and $\alpha_3 = -1$. The first identity is:

$$\sum_{p=1}^{\infty} A_{pq}^{(r)} f_p^{(3)} \frac{\omega_p - \mu}{\omega_p + \omega_n} = \frac{\alpha_3}{\alpha_r} f_q^{(r)} \frac{\omega_{r,q}+\alpha_r\mu}{\omega_{r,q}-\alpha_r\omega_n} - \frac{A_{nq}^{(r)}(\omega_n+\mu)(1+\mu\alpha k)}{\alpha_1\alpha_2 n\omega_n f_n^{(3)}} \tag{31}$$

which is only valid for $r = 1, 2$. For $r = 3$ it is trivial to see we only get the first term. Note that this sum is very similar to one computed in [8], namely (15). Notice that the extra factor does not affect the asymptotics of the integrand of the corresponding contour integral, or the parity of the integrand along the branch cut. Thus we need only worry about the extra residue which occurs at

$p=-n$. This is what gives the second term on the RHS of (31). The first term is the analogue of the RHS of (15) which comes from the residue at $p=-q/\alpha_r$. More explicitly, for the $r=1$ case, one needs to consider the following contour integral

$$\oint \frac{dp}{2\pi i}\frac{2}{\sin(\pi p)}\frac{(-1)^q\sqrt{q}M(0^+)y}{(1-y)}\frac{\sin(\pi y p)}{q^2-y^2p^2}\frac{(\omega_p-\mu)e^{\tau_0(\mu-\omega_p)}}{\omega_p(\omega_p+\omega_n)}\frac{\Gamma_{\mu y}(yp)}{\Gamma_{\mu(1-y)}(-(1-y)p)\Gamma_\mu(p)} \tag{32}$$

where the contour is a large circle centred at the origin. The integrand has simple poles for $p\in\mathbb{N}$, whose residues lead to the sum we want, i.e. the LHS of (31); this is of course by construction. Another simple pole occurs at $p=-q/y$ and leads to the contribution $\frac{f_q^{(1)}}{y}\left(\frac{\omega_{q/y}+\mu}{\omega_{q/y}-\omega_n}\right)$. This is the term that corresponds to the one which gives $f^{(1)}$ in (15), see [8]. The new term comes from the simple pole at $p=-n$ as we have already pointed out. To calculate this, one needs the reflection identities a couple of times, and we find the contribution

$$\frac{A^{(1)}_{nq}e^{2\mu\tau_0}M(0^+)^2}{f_n^{(3)}n\omega_n\alpha_1\alpha_2}. \tag{33}$$

Note that we choose a branch cut on $[i\mu,-i\mu]$, and the line integrals on either side of this cut vanish since the integrand there is odd for $|\mathrm{Im}\,p|<\mu$. Finally, the asymptotics of the integrand are such that the integral on the large circle tends to zero as the radius of the circle tends to infinity; for this one needs the generalisation of Stirling's formula derived in [8]. Piecing all this together we deduce (31) for $r=1$. The $r=2$ version of the identity is easily inferred from the $r=1$ by mapping $y\to 1-y$.

The second identity required is

$$\sum_{q=1}^{\infty}\sum_{r=1}^{3}\frac{\alpha_3}{\alpha_r}\,q\,A^{(r)}_{mq}\,\frac{f_q^{(r)}}{\omega_{r,q}-\alpha_r\omega_n}=\frac{(1+\mu\alpha k)}{\alpha_1\alpha_2\omega_m f_m^{(3)}}\,\delta_{mn}. \tag{34}$$

This sum is very similar to another one evaluated in [8], namely (16). The summand thus differs by a factor of $(\omega_{r,q}+\mu\alpha_r)/(\omega_{r,q}-\alpha_r\omega_n)$, which again does not change the asymptotics or the parity along the branch cut of the corresponding integrand. Also note that the extra factor in the numerator now ensures that the contribution from $q=0$ vanishes on both sides of the branch cut. To prove (34) consider the following contour integral

$$\oint \frac{dq}{2\pi i}\frac{2}{\sin(\pi q)}\frac{(-1)^{m+1}\sqrt{m}\sin(m\pi y)}{q^2-y^2m^2}\frac{qM(0^+)}{\omega_{1,q}-y\omega_n}\frac{e^{\tau_0(\mu+\omega_{q/y})}\,\Gamma_{\mu(1-y)}(-q(1-y)/y)}{\omega_{q/y}\,\Gamma_\mu(-q/y)\Gamma_{\mu y}(q)} \tag{35}$$

where once again the contour is a large circle centred on the origin which we will take to infinity. Note that now there is a branch cut on $[iy\mu,-iy\mu]$. Now, the residues from the simple poles at $q\in\mathbb{N}$ give us the $r=1$ term on the LHS of (34). The simple poles at $q(1-y)/y\in\mathbb{N}$ give the $r=2$ term, and the simple pole at $q=-my$ gives the $r=3$ term. The simple poles of $1/(\omega_{1,q}-y\omega_n)$ at $q=ny$ and of $1/(q^2-y^2m^2)$ at $q=my$, for $n\neq m$, get cancelled by the factor $1/\Gamma_\mu(-q/y)$. For $n=m$ however, we have a simple pole at $q=ny$. To evaluate its residue we need

$$\mathrm{Res}_{q=ny}\left[\frac{1}{(\omega_{1,q}-y\omega_n)(q^2-y^2m^2)\Gamma_\mu(-q/y)}\right]=(-1)^{n+1}\frac{\omega_n}{2ny^2}\Gamma_\mu(n), \tag{36}$$

which after some manipulation, including the use of a reflection identity, gives the following contribution to the above contour integral:

$$-\frac{e^{2\tau_0\mu}M(0^+)^2}{\alpha_1\alpha_2\omega_n f_n^{(3)}}. \tag{37}$$

Observe that the line integrals along either side of the branch cut for $|\mathrm{Im}\, q| < y\mu$ vanish due to the integrand being odd there. The circular integrals around the branch points also vanish as $O(\epsilon^{1/2})$, see [8], where ϵ is the radius from the branch point. Finally, the asymptotics are such that the integral on the circle at infinity vanish [8]. This completes the proof of (34).

Using (31) and (34), it is now a simple matter of some algebra to check that the RHS of (30) is equal to δ_{mn}. This completes the proof of existence of Γ_+^{-1} and as we have already mentioned its uniqueness follows from the fact that it must be both a left and right inverse as Γ_+ is a symmetric matrix.

The author would like to thank Sakura Schafer-Nameki and Aninda Sinha for reading through the manuscript and making helpful comments. The author would also like to thank Malcolm Perry for encouragement to publish the result. The author is supported by EPSRC.

Appendix

Here we summarise some useful definitions and identities.

Open-closed vertex

Recall the definitions of the two functions[6] :

$$\Gamma^I_\mu(z) = e^{-\gamma\omega_{2z}/2}\left(\frac{1}{z}\right)\prod_{n=1}^{\infty}\left(\frac{2n}{\omega_{2z}+\omega_{2n}}\,e^{\omega_{2z}/2n}\right) \tag{38}$$

$$\Gamma^{II}_\mu(z) = e^{-\gamma(\omega_{2z-1}+1)/2}\left(\frac{2}{\omega_{2z-1}+\omega_1}\right)\prod_{n=1}^{\infty}\left(\frac{2n}{\omega_{2z-1}+\omega_{2n+1}}\,e^{(\omega_{2z-1}+1)/2n}\right), \tag{39}$$

which satisfy the crucial reflection identities:

$$\Gamma^I_\mu(z)\Gamma^I_\mu(-z) = -\frac{\pi}{z\sin(\pi z)} \tag{40}$$

$$\Gamma^{II}_\mu(1+z)\Gamma^{II}_\mu(-z) = -\frac{\pi}{\sin(\pi z)}. \tag{41}$$

Note that both functions have simple poles for $z=-n$ where $n\in\mathbb{N}$, and $\Gamma^{II}_\mu(z)$ also has a simple pole at $z=0$. Also $\Gamma^I_\mu(z)$ has a branch cut on $[i\mu/2,-i\mu/2]$, whereas $\Gamma^{II}_\mu(z)$ has a branch cut on $[1/2+i\mu/2, 1/2-i\mu/2]$.

Cubic vertex

Crucial quantities are the momenta of the three strings α_1, α_2 and α_3 which satisfy $\sum_{r=1}^3\alpha_r=0$. We will always choose $\alpha_1=y$, $\alpha_2=1-y$ and hence $\alpha_3=-1$ as was done in [12, 8]. Also $\tau_0=\sum_{r=1}^3\alpha_r\log|\alpha_r|$ and $\alpha=\alpha_1\alpha_2\alpha_3$. The matrices $A^{(r)}_{mn}$ and vector B_m are given by:

$$\begin{aligned}
A^{(1)}_{mn} &= \frac{2}{\pi}(-1)^{m+n+1}\sqrt{mn}\frac{\beta\sin(m\pi\beta)}{n^2-m^2\beta^2}\\
A^{(2)}_{mn} &= \frac{2}{\pi}(-1)^{m+1}\sqrt{mn}\frac{(\beta+1)\sin(m\pi\beta)}{n^2-m^2(\beta+1)^2}\\
A^{(3)}_{mn} &= \delta_{mn}
\end{aligned}$$

[6]Note that the definitions appearing here are slightly different than in [1]. They differ in the denominators of the infinite product, and this only has the effect of rescaling the Gamma functions by a μ dependent factor (i.e. $\prod_{n=1}^{\infty}\frac{\omega_{2n}}{2n}$) which cancels in the expressions for the Neumann matrices. This is to simplify the reflection identities.

$$B_m = \frac{2}{\pi}\frac{\alpha_3}{\alpha_1\alpha_2}(-1)^{m+1}\frac{\sin(m\pi\beta)}{m^{3/2}}\,, \tag{42}$$

where $\beta = \alpha_1/\alpha_3$. Another definition which we use is

$$(U^{(r)})_{mn} = \delta_{mn}\frac{(\omega_{r,m} - \alpha_r\mu)}{m}. \tag{43}$$

The functions $\Gamma^{(r)}_\mu(z)$ are defined as

$$\Gamma^{(r)}_\mu(z) = e^{-\gamma\alpha_r\omega_z}\,\frac{1}{\alpha_r z}\prod_{n=1}^{\infty}\left(\frac{n}{\omega_{r,n}+\alpha_r\omega_z}\,e^{\frac{\alpha_r\omega_z}{n}}\right) \tag{44}$$

$$= \Gamma^I_{2\mu\alpha_r}(\alpha_r z) \tag{45}$$

and $\Gamma_\mu(z) \equiv \Gamma^I_{2\mu}(z)$. They have simple poles for $-z \in \mathbb{N}$ and branch cuts on $[i\mu, -i\mu]$. We have the reflection identities

$$\Gamma^{(r)}_\mu(z)\Gamma^{(r)}_\mu(-z) = -\frac{\pi}{\alpha_r z\sin(\pi\alpha_r z)}. \tag{46}$$

The constant $M(0^+)$ is given by the function

$$M(z) = \frac{\Gamma_\mu(z)z}{\Gamma^{(1)}_\mu(z)\alpha_1 z\,\Gamma^{(2)}_\mu(z)\alpha_2 z}, \tag{47}$$

and $M(0^+) = \lim_{z\to 0^+} M(z)$.

References

[1] J. Lucietti, S. Schafer-Nameki, A. Sinha, *On the exact open-closed vertex in plane-wave light-cone string field theory,* Phys. Rev. D **69**, 086005 (2004); hep-th/0311231.

[2] R. R. Metsaev, *Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*, Nucl. Phys. B **625**, 70 (2002); hep-th/0112044.

[3] R. R. Metsaev, A. A. Tseytlin, *Exactly solvable model of superstring in plane wave Ramond-Ramond background,* Phys. Rev. D **65**, 126004 (2002); hep-th/0202109.

[4] D. Berenstein, J. M. Maldacena, H. Nastase, *Strings in flat space and pp waves from $N = 4$ super Yang Mills,* JHEP **0204**, 013 (2002); hep-th/0202021.

[5] O. Bergman, M. R. Gaberdiel, M. B. Green, *D-brane interactions in type IIB plane-wave background,* JHEP **0303**, 002 (2003); hep-th/0205183.

[6] M. R. Gaberdiel, M. B. Green, *The D-instanton and other supersymmetric D-branes in IIB plane-wave string theory,* Annals Phys. **307**, 147 (2003); hep-th/0211122.

[7] M. R. Gaberdiel, M. B. Green, S. Schafer-Nameki and A. Sinha, *Oblique and curved D-branes in IIB plane-wave string theory,* JHEP **0310** (2003) 052; hep-th/0306056.

[8] J. Lucietti, S. Schafer-Nameki, A. Sinha, *On the plane-wave cubic vertex,* To appear in Phys. Rev. D; hep-th/0402185

[9] M. B. Green, J. H. Schwarz, *Superstring Field Theory*, Nucl. Phys. B **243**, 475 (1984).

[10] M. B. Green, J. H. Schwarz, *Superstring Interactions,* Nucl. Phys. B **218**, 43 (1983).

[11] J. H. Schwarz, *The generalized Stieltjes transform and its inverse*; math-ph/0405050.

[12] Y. H. He, J. H. Schwarz, M. Spradlin, A. Volovich, *Explicit formulas for Neumann coefficients in the plane-wave geometry,* Phys. Rev. D **67**, 086005 (2003); hep-th/0211198.

[13] J. H. Schwarz, *The three-string vertex for a plane-wave background*; hep-th/0312283.

[14] J. Gomis, S. Moriyama and J. w. Park, *Open + closed string field theory from gauge fields*, Nucl. Phys. B **678** (2004) 101; hep-th/0305264.

[15] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background*, Phys. Rev. D **66** (2002) 086004; hep-th/0204146.

[16] M. Spradlin and A. Volovich, *Superstring interactions in a pp-wave background. II,* JHEP **0301** (2003) 036; hep-th/0206073.

[17] J. H. Schwarz, *Comments on superstring interactions in a plane-wave background,* JHEP **0209**, 058 (2002); hep-th/0208179.

[18] A. Pankiewicz, *More comments on superstring interactions in the pp-wave background*, JHEP **0209** (2002) 056; hep-th/0208209.